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A Stochastic Volatility Approximation for a Tick-By-Tick Price Model with Mean-Field Interaction

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A stochastic volatility approximation for a tick-by-tick price model with mean-field interaction

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Abstract

We consider a tick-by-tick model of price formation, in which buy and sell orders are modeled as self-exciting point processes (Hawkes process), similar to the one in [El Euch, Fukasawa, Rosenbaum, *The microstructural foundations of leverage effect and rough volatility*, Finance and Stochastics, 2018]. We adopt an agent based approach by studying the aggregation of a large number of these point processes, mutually interacting in a mean-field sense.

The financial interpretation is that of an asset on which several labeled agents place buy and sell orders following these point processes, influencing the price. The mean-field interaction introduces positive correlations between order volumes coming from different agents that reflect features of real markets such as herd behavior and contagion. When the large scale limit of the aggregated asset price is computed, if parameters are set to a critical value, a singular phenomenon occurs: the aggregated model converges to a stochastic volatility model with leverage effect and faster-than-linear mean reversion of the volatility process.

The faster-than-linear mean reversion of the volatility process is supported by econometric evidence, and we have linked it in [Dai Pra, Pigato, *Multi-scaling of moments in stochastic volatility models*, Stochastic Processes and their Applications, 2015] to the observed multifractal behavior of assets prices and market indices. This seems connected to the Statistical Physics perspective that expects anomalous scaling properties to arise in the critical regime.

Keywords: Stochastic Volatility, Hawkes processes, multifractality, mean-field, non-linearity, criticality

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1 Introduction

We consider a tick-by-tick model of price formation, in which price variations due to buy and sell orders are modeled as self-exciting point processes (Hawkes process) similar to the ones in [24, 25]. This model encodes well-documented and understood behaviors of markets at the microscopic scale. In this paper, we study the aggregation of a large number of these point processes, self-exciting with short memory, supposing that they are mutually interacting in a mean-field sense. The financial interpretation is that of an asset whose price variations are the aggregation, in a suitable sense, of the variations due to the participation to the process of price formation of several labeled agents, with positive correlations between price variations due to orders from different agents, that reflect features of real markets such as herd behavior and contagion [8, 30].

When the large scale limit of the aggregated price is computed, a singular phenomenon occurs: the system displays a special scaling if the model's parameters are set to a critical value. This seems to be inherent in these models based on Hawkes processes, when considering that the near instability condition in [25, 51, 10] corresponds to the criticality condition we assume in this paper. We find that the price

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dynamics constructed via this aggregation procedure converges, in the large scale limit, in the critical regime, to a stochastic volatility model with leverage effect (as in [25]) and faster-than-linear mean reversion of the volatility process.

The faster-than-linear mean reversion of the volatility process is supported by econometric evidence [5]. Moreover, models with quadratic mean reversion in the volatility process have been used for option pricing (e.g. on bitcoins) and connected issues [46, 42], and it is the drift form in the volatility function of the 3/2-model [6]. In [19], we have linked faster-than-linear mean reversion of the volatility process to the observed multifractal behavior of asset and index prices [39].

This seems connected to the Statistical Physics perspective, that expects anomalous scaling properties to arise in the critical regime. The same paradigm has been used to explain scaling phenomena in many different fields, including natural and social sciences, see e.g. [49] for a comprehensive review and [15] for applications to market models.

Background - literature review. A relevant question in the financial literature is understanding to what extent macroscopic financial dynamics can be derived as limits of microscopic, possibly agent-based models [14, 18]. In [24, 25, 38, 3, 40], the microscopic dynamics of prices is modeled with Hawkes processes, which are self-exciting point processes. In particular, in the recent work [25], a rough version of the Heston model is derived as high-frequency limit of a market model in which price variations due to buy and sell orders are modeled as Hawkes processes with long memory. In addition to roughness, this provides a microscopic foundation of the leverage effect.

In this paper, we consider the aggregation of a large number of these microscopic Hawkes dynamics, assuming a mean-field interaction. This type of interactions have been widely studied in financial mathematics and other disciplines [48, 47]. In [51], scaling limits for a large number of Hawkes processes as the ones in [25], with mean-field interactions, are established. This extends in a certain direction the fundamental study on large systems of Hawkes processes with mean-field interaction in [22]. We also mention [31], where the 3/2 model is derived as a stochastic volatility approximation (with faster than linear mean reversion) for a large-basket price-weighted index, where the components of the basket have diffusive and jumps parts, and interact in a mean-field sense.

In our model, we find that in the large-scale continuous-time limit the behavior displays a special scaling if the model's parameters are set to a critical value. In this case, the limit behavior is that of a stochastic volatility model with quadratic speed of mean reversion of the volatility process. We refer the reader to [5] for econometric evidence of superlinear mean reversion of volatility and [46, 42, 12, 20] for applications of this type of volatility process in option pricing and related issues. See also [6] and references therein for the 3/2 model, which has a similar drift of volatility but a different volatility of volatility compared to our limit model.

As we mentioned, several works [25, 51, 10] hint at the fact that one should look at these Hawkes processes-based models close to a critical point. The question of why many real complex systems self-organize at or close to the critical point is however, to a large extent, yet to be understood at a rigorous level. The notion of self-organized criticality was proposed in the fundamental paper [4]. In few models this phenomenon has allowed a rigorous analysis; we mention the sandpile models [2] and Curie-Weiss type models [13]. Self organized criticality is often associated with the existence of multiple scaling regimes (multifractality), see [43] for an account. Multifractality is a well extablished stylized fact of market indices, stocks, commodities, exchange rates, interest rates and other financial time series [39]. In connection to these facts, in [19], we show that in a certain class of Levy-driven volatility models, multifractality (in the form of multiscaling of moments, see next Remark 2.6) is possible only if the mean-reverting drift function is superlinear. We remark that, in the model we analyse in the present paper, the nonlinearity of the mean reversion is an example of *anomalous* critical scaling. If the parameters were set below the critical point, then the price process would converge to a model with linear mean reversion, but that would become trivial as the critical point is approached, in the sense that the mean reversion would vanish. At the critical point the time needs to be rescaled to obtain a nontrivial limit, and linearity is lost in the limit. This is close in spirit to anomalous fluctuation theorems in models motivated by Statistical Mechanics, where non Gaussian distributions appear in the limit [17, 7].

Finally, leverage effect refers to the widely documented negative correlation between observed changes in stock returns and volatility [26], see also [25, 41] and references therein. It is often incorporated in models for financial price dynamics through a negative correlation between the noises driving price and volatility, see [44, 34, 36, 50]. Leverage effect can be explained by the fact that when an asset price decreases, the ratio of the company's debt with respect to the equity value becomes larger, and as a consequence volatility increases. Other "macroscopic" explanations from from financial economics have been proposed [9, 16, 33, 28], as well as "psycological" explanations based on asymmetric responses by investors [32]. Here, as in [25], leverage effect observed on macroscopic quantities is a product of microstructural features, with in addition an explicit, stylized representation of interactions between agents.

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Outline. In Section 2 we describe our model, state the main result of the paper and discuss it. Section 3 contains proofs and technical material, while in the appendix we collect known results on which our proofs hinge and some auxiliary computations.

2 Model description and main results

We consider a market where N agents are placing orders for a given asset. The counting processes N_i^+ and N_i^-, for $i = 1, \dots, N$, are Hawkes processes and represent variations (upward and downward, respectively) in the price due to orders (buy and sell, respectively) placed by agent i. We refer the reader to [29, 35, 23] and the references therein for more information on impact of buy and sell orders on price. The contribution of agent i to the asset log-price is therefore the sum of its positive and negative price variations

$$P_i = N_i^+ - N_i^-. \tag{1}$$

We also interpret this model as a decomposition over individual agents of the model in [25, 24]. See next Remark 2.5) for an alternative interpretation in the case of market indices or portfolios.

We consider in this paper the function $\varphi(t) = e^{-\alpha t}$, ruling the memory of the process. Note that more general choices of $\varphi : [0, +\infty) \to [0, +\infty)$ have been considered, see for example [25]; however, our analysis relies on this choice. Let us introduce two sequences $\nu_N^\pm$ of σ-finite measures on $[0, +\infty)$, and define

$$X_i^\pm(t) := \int_{[0,t]} \varphi(t-s) \left[dN_i^\pm(s) + \nu_N^\pm(ds) \right]. \tag{2}$$

We then consider the empirical means

$$m_N^\pm(t) := \frac{1}{N} \sum_{i=1}^N X_i^\pm(t).$$

We will assume that the point processes $(N_i^\pm)_{i=1}^N$ are conditionally independent given $(X_i^\pm)_{i=1}^N$. Each "upward" jump process N_i^+ has a stochastic intensity given by

$$\lambda_N^+(t) := f(m_N^+(t) + \beta\gamma m_N^-(t)), \tag{3}$$

while the"downward" jumps have stochastic intensity

$$\lambda_N^-(t) := f(\gamma m_N^+(t) + (1 + (\beta - 1)\gamma) m_N^-(t)), \tag{4}$$

where f is a given increasing, concave function of class $\mathcal{C}^3$, with $f(0) = 0$, $f'(0) > 0$, $f''(0) < 0$ with f', f'' and f''' bounded, and $\beta \geq 1$, $\gamma \in [0, 1]$ are parameters ruling the self and cross excitation of upward and downward jumps (see Remarks 2.2 and 2.3).

The processes $(N_i^\pm)_{i=1}^n$ are called Hawkes processes. Note that the intensity is an increasing function of the mean $m_N^\pm$ and is the same for each agent, which should reflect contagion and homogeneity of the model. The *mean-field* interaction models the fact that buy orders, with consequent increase in the log-price (and viceversa, sell orders with consequent decrease in the log-price) are exciting orders of the same type in other labeled agents. Moreover, the processes $X_i^{\pm}$ that are responsible for the self-excitation of the $N_i^{\pm}$, are also driven by a common, external signal modeled by the measure $\nu_N^{\pm}$, representing a "baseline" volatility. In what follows we will assume that

$$\nu_N^{\pm}(dt) = a_N^{\pm}\delta_0(dt) + b_N^{\pm}dt,$$

where a_N, b_N are positive constants satisfying

$$\lim_{N\to+\infty} \sqrt{N}a_N^{\pm} = a^{\pm} \in (0,+\infty), \qquad \lim_{N\to+\infty} \sqrt{N}b_N^{\pm} = b^{\pm} \in (0,+\infty). \tag{5}$$

Note that a_N is responsible for the value at time $t=0$ of the stochastic intensities $\lambda_N^{\pm}$, while b_N represents a stationary external signal. More irregular choices for ν_N are also possible.

We now introduce the rescaled log-price process

$$\Pi_N(t) = \frac{1}{\sqrt{N}}\sum_{i=1}^{N} P_i(\sqrt{N}t). \tag{6}$$

Our main result is the following (see next Theorem 3.2 for a more detailed version of the theorem).

Theorem 2.1. *Suppose the criticality condition*

$$\alpha = f'(0)(1+\beta\gamma) \tag{7}$$

holds. The process Π_N converges in distribution in $(0,T]$ to the process π, where (π, y) solves the SDE

$$\begin{aligned} d\pi(t) &= \beta_\pi dt + \sigma_\pi\sqrt{f'(0)y(t)}dW(t) \\ dy(t) &= (\beta_y + \alpha_y f''(0)y^2(t))dt + \sigma_y\sqrt{f'(0)y(t)}dB(t) \end{aligned} \tag{8}$$

with initial condition

$$\pi(0) = \frac{1-\gamma}{\gamma(1+\beta)}(a^+ - a^-), \quad y(0) = \left(\frac{2}{1+\beta} + \frac{2\beta}{1+\beta}\gamma\right)a^+ + \left(\frac{2\beta}{1+\beta}(1-\gamma) + 2\beta\gamma\right)a^-,$$

where the constants in the dynamics depend on $\beta, \gamma, b^{\pm}$ as below

$$\beta_\pi = \frac{1-\gamma}{\gamma(1+\beta)}(b^+ - b^-); \quad \sigma_\pi = \sqrt{2}\frac{1+\beta\gamma}{\gamma(1+\beta)};$$

$$\beta_y = \left(\frac{2}{1+\beta} + \frac{2\beta}{1+\beta}\gamma\right)b^+ + \left(\frac{2\beta}{1+\beta}(1-\gamma) + 2\beta\gamma\right)b^-; \quad \alpha_y = \frac{1}{2}(1+\beta\gamma); \quad \sigma_y = \frac{\sqrt{(1+\beta^2)(1+\beta\gamma)}}{1+\beta}$$

and W, B are standard Brownian motions, with correlation

$$d\langle B, W\rangle_t = \frac{(1+\beta)\gamma}{1+\beta\gamma}\frac{(1-\beta)}{\sqrt{2(1+\beta^2)}}dt. \tag{9}$$

Remark 2.2. Note that (3) and (4) express the jump intensity as a concave function of a functional of past jumps. For instance, referring to (3), the term $m_N^+(t)$ can be interpreted as the impact of past jump upwards on λ_N^+, and $\beta\gamma m_N^-(t)$ as the impact of past jump downwards on λ_N^+. The condition $\gamma \le 1$ corresponds to the requirement "past jump upwards (resp. downwards) impact more on λ_N^+ than on λ_N^- (resp. more on λ_N^- than on λ_N^+)", which we need to encode herd behavior in our model. Indeed,

$$m_N^+ \ge \gamma m_N^+ \text{ and } (1+(\beta-1)\gamma)m_N^- \ge \beta\gamma m_N^-,$$

with equality for $\gamma = 1$. The condition $\beta \ge 1$ is related to the higher impact of the jump downwards, in the following sense: "the impact of jump downwards on λ_N^+ (resp. λ_N^-) is greater than the impact of jump upwards on λ_N^- (resp. λ_N^+). Indeed the coefficient $\beta\gamma$ of $m_N^-(t)$ in (3) is greater than the coefficient γ of $m_N^+(t)$ in (4), and the coefficient $(1+(\beta-1)\gamma)$ of m_N^- in (4) is greater than the coefficient 1 of m_N^+ in (3), with equality for $\beta = 1$. This condition seems natural in a financial context due to the higher "excitatory power" of downard price movements, i.e., the fact that investors react more to a decrease than to an increase in the price; cf. also the explanation in [25]. This corresponds to $\beta \geq 1$, and gives rise to leverage effect in the limit system in the sense of Remark 2.4. Note also that the requirements on the impact of jump upwards and downwards lead naturally to introduce at least two parameters in the model, possibly with a different combination from the one we have chosen.

Remark 2.3. In our setting, equations (3) and (4) encompass the fact that the effect on present jumps intensity of compounded past jumps averaged over all the agents is not linear, but mediated by the increasing concave function f, meaning that a saturation effect is at play, with larger past jumps producing a stronger intensity of present jumps, with a decreasing marginal dependence. Note that the y process in (8) is mean reverting, since $f''(0) < 0$.

Remark 2.4. [Leverage effect] Since $\gamma \in [0,1]$ and $\beta \geq 1$, it always holds that $(1+\beta)\gamma/(1+\beta\gamma) \in [0,1]$ and $\langle B, W\rangle \in (-1/\sqrt{2}, 0]$. When $\beta \to \infty$, we have $\langle B, W\rangle \to -1/\sqrt{2}$. Recall that in [25], the correlation in the limit model is $(1-\beta)/\sqrt{2(1+\beta^2)}$, which we obtain here with $\gamma = 1$. The fact that $\langle B, W\rangle \leq 0$ means that this model displays a negative relationship between changes in stock returns and in volatility, the so called leverage effect. This is a reflection of the fact that $\beta \geq 1$, representing, as seen in Remark 2.2, the fact that jumps downwards are overall more excitatory than jumps upwards.

Remark 2.5. [Index and portfolio dynamics] A different possible interpretation of the aggregated price in (6) and its limit $(\pi_t)_t$, is that of an index or portfolio composed of $i = 1, \dots, N$ assets. The variations of the index are given by an average of the variations of the single components of the index, each represented by the sum of its positive and negative "micro" variations. The single assets have dynamics as in [25, 24], they are self exciting and are exciting each other, with jumps in one price exciting jumps in all the others through a mean-field interaction. The diffusive dynamics is seen at a slower time scale than the order book time scale where we see the jumps in the prices.

Remark 2.6. [Multifractality - Multiscaling of moments] Let $(X_t)_{t\geq 0}$ be a continuous-time martingale, having stationary increments; in financial applications this could be identified with the de-trended *log-price* of an asset, or the log-price with respect to the martingale measure used for pricing derivatives. We say that multiscaling of moments occurs if $\mathbb{E}(|X_{t+h} - X_t|^q)$ scales, in the limit as $h \downarrow 0$, as $h^{A(q)}$, with $A(q) = \frac{q}{2}$ for small q but $A(q) < \frac{q}{2}$ beyond a certain threshold. The presence of multiscaling in financial time series is well established and various models have been proposed to capture it, most notably multi-fractal processes [11], see also [52, 45, 1]. In [19] we have shown that in a stochastic volatility model whose volatility V_t solves the stochastic differential equation

$$dV_t = -f(V_t)dt + dL_t, \tag{10}$$

with L a *Levy process*, multiscaling occurs if and only if the characteristic measure of L has power law tails at infinity, and f is a *superlinear* mean reversion function, a fact that is supported e.g. by the empirical findings in [5]. Therefore, the result in Theorem 2.1 is a possible explanation of the origin of the superlinear mean reversion, while jumps may be produced by exogenous shocks.

3 Proofs

The proof of Theorem 2.1 is divided in two steps. We first identify a *microscopic* volatility process $Y_N(t)$, and we show it converges in distribution to a solution of the second equation in (8). We use martingale methods for this convergence; the main difficulty is to show that certain terms in the dynamics of Y_N vanish as $N \to +\infty$, which is essential to obtain a closed equation for the limit of Y_N. As it is customary in critical dynamics, there is one divergent term (as $N \to +\infty$) in the dynamics of Y_N, which however vanishes if the criticality condition (7) holds. In the second step we prove the convergence of the price process. This turns out to be more difficult, as the divergent term in the price dynamics does not disappear at criticality. This type of problem appears in perturbation theory of operators (see [21] for an application to a mean field model). Here we circumvent the problem by applying a theorem on *collapsing processes* due to Comets & Eisele [17], which may be seen as the probabilistic counterpart of a perturbation theory argument. Theorem A.1, Theorem A.2, Lemma B.1 and Theorem B.2, which we use along the proof, are given in the appendix.

3.1 Preliminaries

Using our specific choice $\varphi(t) = e^{-\alpha t}$, by Fubini's Theorem and the fact that $\phi(t) = -\frac{1}{\alpha}\frac{d}{dt}\phi(t)$:

$$\begin{aligned}\int_0^t X_i^{\pm}(s)ds &= \int_0^t \int_{[0,s]} \varphi(s-r)\left[dN_i^{\pm}(r) + \nu_N^{\pm}(dr)\right] ds = \int_{[0,t]} \left(\int_r^t \varphi(s-r)ds\right)\left[dN_i^{\pm}(r) + \nu_N^{\pm}(dr)\right] \\ &= \int_{[0,t]} \frac{1}{\alpha}\left(1-\varphi(t-r)\right)\left[dN_i^{\pm}(r) + \nu_N^{\pm}(dr)\right] = \frac{1}{\alpha}N_i^{\pm}(t) + \frac{1}{\alpha}\nu_N^{\pm}([0,t]) - \frac{1}{\alpha}X_i^{\pm}(t),\end{aligned}$$

and therefore

$$X_i^{\pm}(t) = -\alpha\int_0^t X_i^{\pm}(s)ds + N_i^{\pm}(t) + \nu_N^{\pm}([0,t]). \tag{11}$$

Note that this computation relies on our specific functional choice for ϕ, and not only on the memory.

We now give a representation of the processes $X_i^{\pm}(t)$ in terms of a family $n_i^{+}(ds,du)$, $n_i^{-}(ds,du)$ of independent Poisson Random Measures on $[0,+\infty)^2$ with intensity measure $ds\,du$. We can write

$$X_i^{\pm}(t) = -\alpha\int_0^t X_i^{\pm}(s)ds + \int_{[0,t]\times[0,+\infty)} \mathbf{1}_{[0,\lambda_N^{\pm}(s^-))}(u)n_i^{\pm}(ds,du) + a_N^{\pm} + b_N^{\pm}t, \tag{12}$$

which can be aggregated to

$$m_N^{\pm}(t) = -\alpha\int_0^t m_N^{\pm}(s)ds + \int_{[0,t]\times[0,+\infty)} \mathbf{1}_{[0,\lambda_N^{\pm}(s^-))}(u)\frac{1}{N}\sum_{i=1}^N n_i^{\pm}(ds,du) + a_N^{\pm} + b_N^{\pm}t. \tag{13}$$

Using then the compensated Poisson Random Measures $\tilde{n}_i^{\pm}(ds,du) = n_i^{\pm}(ds,du) - ds\,du$ we obtain the semimartingale representation

$$m_N^{\pm}(t) = -\alpha\int_0^t m_N^{\pm}(s)ds + \int_0^t \lambda_N^{\pm}(s)ds + a_N^{\pm} + b_N^{\pm}t + \int_{[0,t]\times[0,+\infty)} \mathbf{1}_{[0,\lambda_N^{\pm}(s^-))}(u)\frac{1}{N}\sum_{i=1}^N \tilde{n}_i^{\pm}(ds,du). \tag{14}$$

Note that

$$M_N^{\pm}(t) := \int_{[0,t]\times[0,+\infty)} \mathbf{1}_{[0,\lambda^{\pm}(s^-))}(u)\frac{1}{N}\sum_{i=1}^N \tilde{n}_i^{\pm}(ds,du)$$

are orthogonal martingales with quadratic variations $\frac{1}{N}\int_0^t \lambda_N^{\pm}(s)ds$.

3.2 Definition and convergence of the volatility process

We now define

$$Y_N(t) = \frac{\sqrt{N}}{2}\left[\left(\frac{2}{1+\beta} + \frac{2\beta}{1+\beta}\gamma\right)m_N^{+}(\sqrt{N}t) + \left(\frac{2\beta}{1+\beta}(1-\gamma) + 2\beta\gamma\right)m_N^{-}(\sqrt{N}t)\right]$$

and

$$Z_N(t) = \frac{(1-\gamma)\sqrt{N}}{2}\left[m_N^{+}(\sqrt{N}t) - m_N^{-}(\sqrt{N}t)\right].$$

Note that

$$\lambda_N^{+}(t) = f\left(\frac{1}{\sqrt{N}}Y_N(t/\sqrt{N}) + \frac{1}{\sqrt{N}}\frac{2\beta}{1+\beta}Z_N(t/\sqrt{N})\right)$$

and

$$\lambda_N^-(t) = f\left(\frac{1}{\sqrt{N}} Y_N(t/\sqrt{N}) - \frac{1}{\sqrt{N}}\frac{2}{1+\beta} Z_N(t/\sqrt{N})\right)$$

Moreover, observing that $Y_N \geq 0$ and that the arguments of f in the definition of $\lambda_N^{\pm}$ are nonnegative, we obtain the bound

$$|Z_N(t)| \leq \frac{1+\beta}{2} Y_N(t). \tag{15}$$

We are going to see that the process Y_N has a nontrivial limit as $N \to +\infty$, while Z_N converges to zero in a suitable sense. We begin by studying Z_N. Using (13) we obtain

$$\begin{aligned} Z_N(t) &= E_N + F_N t - \alpha\sqrt{N}\int_0^t Z_N(s)ds \\ &+ \frac{1-\gamma}{2\sqrt{N}}\int_{[0,\sqrt{N}t]} \mathbf{1}_{[0,\lambda_N^+(s))}(u)\sum_{i=1}^N n_i^+(ds,du) - \frac{1-\gamma}{2\sqrt{N}}\int_{[0,\sqrt{N}t]} \mathbf{1}_{[0,\lambda_N^-(s))}(u)\sum_{i=1}^N n_i^-(ds,du), \end{aligned} \tag{16}$$

where

$$E_N = \frac{1-\gamma}{2}\sqrt{N}[a_N^+ - a_N^-] \quad F_N = \frac{1-\gamma}{2}\sqrt{N}[b_N^+ - b_N^-].$$

Now a truncation argument is needed for later use. Note that (16) can be seen as an equation for Z_N *given* the process Y_N. Actually, for a given process $\tilde{Y}_N$ and after having extended by setting $f(x) = 0$ for $x < 0$, the corresponding solution $\tilde{Z}_N$ can be obtained pathwise, i.e. given the jumps of the processes $n_i^{\pm}$. In particular, for a fixed $h > 0$, we choose $\tilde{Y}_N(t) := Y_N(t) \wedge h$. Note that $\tilde{Z}_N(t) = Z_N(t)$ up to the stopping time

$$\tau_h^N := \inf\{t > 0 : Y_N(t) > h\}.$$

We have

$$\begin{aligned} \tilde{Z}_N(t) &= E_N + F_N t - \alpha\sqrt{N}\int_0^t \tilde{Z}_N(s)ds \\ &+ \frac{1-\gamma}{2\sqrt{N}}\int_{[0,\sqrt{N}t]} \mathbf{1}_{[0,\tilde{\lambda}_N^+(s))}(u)\sum_{i=1}^N n_i^+(ds,du) - \frac{1-\gamma}{2\sqrt{N}}\int_{[0,\sqrt{N}t]} \mathbf{1}_{[0,\tilde{\lambda}_N^-(s))}(u)\sum_{i=1}^N n_i^-(ds,du), \end{aligned} \tag{17}$$

where

$$\tilde{\lambda}_N^+(t) = f\left(\frac{1}{\sqrt{N}} \tilde{Y}_N(t/\sqrt{N}) + \frac{1}{\sqrt{N}}\frac{2\beta}{1+\beta} \tilde{Z}_N(t/\sqrt{N})\right)$$

and

$$\tilde{\lambda}_N^-(t) = f\left(\frac{1}{\sqrt{N}} \tilde{Y}_N(t/\sqrt{N}) - \frac{1}{\sqrt{N}}\frac{2}{1+\beta} \tilde{Z}_N(t/\sqrt{N})\right)$$

It follows that

$$\begin{aligned} \tilde{Z}_N^2(t) &= E_N^2 + 2F_N\int_0^t \tilde{Z}_N(s)ds - 2\alpha\sqrt{N}\int_0^t \tilde{Z}_N^2(s)ds \\ &+ \int_{[0,\sqrt{N}t]} \mathbf{1}_{[0,\tilde{\lambda}_N^+(s))}(u)\left[\left(\tilde{Z}_N\left(\frac{s^-}{\sqrt{N}}\right) + \frac{1-\gamma}{2\sqrt{N}}\right)^2 - \tilde{Z}_N^2\left(\frac{s^-}{\sqrt{N}}\right)\right]\sum_{i=1}^N n_i^+(ds,du) \\ &+ \int_{[0,\sqrt{N}t]} \mathbf{1}_{[0,\tilde{\lambda}_N^-(s))}(u)\left[\left(\tilde{Z}_N\left(\frac{s^-}{\sqrt{N}}\right) - \frac{1-\gamma}{2\sqrt{N}}\right)^2 - \tilde{Z}_N^2\left(\frac{s^-}{\sqrt{N}}\right)\right]\sum_{i=1}^N n_i^-(ds,du) \end{aligned} \tag{18}$$

Using then the compensated Poisson Random Measures $\tilde{n}_i^{\pm}(ds,du) = n_i^{\pm}(ds,du) - ds\,du$ as above we obtain

$$\begin{aligned}
\tilde{Z}_N^2(t) &= E_N^2 + 2F_N \int_0^t \tilde{Z}_N(s)ds - 2\alpha\sqrt{N}\int_0^t \tilde{Z}_N^2(s)ds \\
&+ N\int_0^{\sqrt{N}t} \left[\frac{(1-\gamma)^2}{4N} + \frac{1-\gamma}{\sqrt{N}}\tilde{Z}_N\left(\frac{s}{\sqrt{N}}\right)\right] f\left(\frac{1}{\sqrt{N}}\tilde{Y}_N\left(\frac{s}{\sqrt{N}}\right) + \frac{1}{\sqrt{N}}\frac{2\beta}{1+\beta}\tilde{Z}_N\left(\frac{s}{\sqrt{N}}\right)\right)ds \\
&+ N\int_0^{\sqrt{N}t} \left[\frac{(1-\gamma)^2}{4N} - \frac{1-\gamma}{\sqrt{N}}\tilde{Z}_N\left(\frac{s}{\sqrt{N}}\right)\right] f\left(\frac{1}{\sqrt{N}}\tilde{Y}_N\left(\frac{s}{\sqrt{N}}\right) - \frac{1}{\sqrt{N}}\frac{2}{1+\beta}\tilde{Z}_N\left(\frac{s}{\sqrt{N}}\right)\right)ds \\
&+ M_N^{\tilde{Z}^2}(t) \\
&= E_N^2 + 2F_N \int_0^t \tilde{Z}_N(s)ds - 2\alpha\sqrt{N}\int_0^t \tilde{Z}_N^2(s)ds \\
&+ N^{3/2}\int_0^t \left[\frac{(1-\gamma)^2}{4N} + \frac{1-\gamma}{\sqrt{N}}\tilde{Z}_N(s)\right] f\left(\frac{1}{\sqrt{N}}\tilde{Y}_N(s) + \frac{1}{\sqrt{N}}\frac{2\beta}{1+\beta}\tilde{Z}_N(s)\right)ds \\
&+ N^{3/2}\int_0^t \left[\frac{(1-\gamma)^2}{4N} - \frac{1-\gamma}{\sqrt{N}}\tilde{Z}_N(s)\right] f\left(\frac{1}{\sqrt{N}}\tilde{Y}_N(s) - \frac{1}{\sqrt{N}}\frac{2}{1+\beta}\tilde{Z}_N(s)\right)ds \\
&+ M_N^{\tilde{Z}^2}(t),
\end{aligned} \tag{19}$$

where $M_N^{\tilde{Z}^2}$ is a mean zero martingale. Taking the expectation in (19) we obtain

$$\begin{aligned}
\frac{d}{dt}\mathbb{E}\left[\tilde{Z}_N^2(t)\right] &= 2F_N\mathbb{E}\left[\tilde{Z}_N(t)\right] - 2\alpha\sqrt{N}\mathbb{E}\left[\tilde{Z}_N^2(t)\right] \\
&+ N^{3/2}\mathbb{E}\left[\left[\frac{(1-\gamma)^2}{4N} + \frac{1-\gamma}{\sqrt{N}}\tilde{Z}_N(t)\right] f\left(\frac{1}{\sqrt{N}}\tilde{Y}_N(t) + \frac{1}{\sqrt{N}}\frac{2\beta}{1+\beta}\tilde{Z}_N(s)\right)\right] \\
&+ N^{3/2}\mathbb{E}\left[\left[\frac{(1-\gamma)^2}{4N} - \frac{1-\gamma}{\sqrt{N}}\tilde{Z}_N(t)\right] f\left(\frac{1}{\sqrt{N}}\tilde{Y}_N(t) - \frac{1}{\sqrt{N}}\frac{2}{1+\beta}\tilde{Z}_N(t)\right)\right]
\end{aligned} \tag{20}$$

Using the fact that f is concave for $x > 0$ we have that $f(x) \le f'(0)|x|$ for all $x \in \mathbb{R}$. So

$$\begin{aligned}
&\frac{(1-\gamma)^2}{4N}\left[f\left(\frac{1}{\sqrt{N}}\tilde{Y}_N(t) + \frac{1}{\sqrt{N}}\frac{2\beta}{1+\beta}\tilde{Z}_N(t)\right) + f\left(\frac{1}{\sqrt{N}}\tilde{Y}_N(t) - \frac{1}{\sqrt{N}}\frac{2}{1+\beta}\tilde{Z}_N(t)\right)\right] \\
&\qquad \le \frac{(1-\gamma)^2}{2N^{3/2}}f'(0)\left[\tilde{Y}_N(t) + 2|\tilde{Z}_N(t)|\right].
\end{aligned}$$

Moreover f is Lipschitz with constant $f'(0)$, so

$$\begin{aligned}
&\frac{1-\gamma}{\sqrt{N}}\tilde{Z}_N(t)\left[f\left(\frac{1}{\sqrt{N}}\tilde{Y}_N(t) + \frac{1}{\sqrt{N}}\frac{2\beta}{1+\beta}\tilde{Z}_N(t)\right) - f\left(\frac{1}{\sqrt{N}}\tilde{Y}_N(t) - \frac{1}{\sqrt{N}}\frac{2}{1+\beta}\tilde{Z}_N(t)\right)\right] \\
&\quad \le \frac{1-\gamma}{\sqrt{N}}|\tilde{Z}_N(t)|\left|f\left(\frac{1}{\sqrt{N}}\tilde{Y}_N(t) + \frac{1}{\sqrt{N}}\frac{2\beta}{1+\beta}\tilde{Z}_N(t)\right) - f\left(\frac{1}{\sqrt{N}}\tilde{Y}_N(t) - \frac{1}{\sqrt{N}}\frac{2}{1+\beta}\tilde{Z}_N(t)\right)\right| \\
&\quad \le \frac{2f'(0)(1-\gamma)}{N}\tilde{Z}_N^2(t).
\end{aligned}$$

Summing all up we obtain

$$\begin{aligned}
\frac{d}{dt}\mathbb{E}\left[\tilde{Z}_N^2(t)\right] &\le [2F_N + (1-\gamma)^2 f'(0)]\mathbb{E}\left[|\tilde{Z}_N(t)|\right] \\
&- 2[\alpha - (1-\gamma)f'(0)]\sqrt{N}\mathbb{E}\left[\tilde{Z}_N^2(t)\right] + \frac{(1-\gamma)^2}{2}f'(0)\tilde{Y}_N(t).
\end{aligned} \tag{21}$$

Using the fact that the sequence F_N is bounded (see (5)), that $\tilde{Y}_N(t) \le h$ and using the Cauchy-Schwartz inequality, we have that $x_N(t) := \mathbb{E}\left[\tilde{Z}_N^2(t)\right]$ satisfies the differential inequality for suitable constants

$A, B > 0$:

$$\frac{d}{dt} x_N(t) \leq A + B\sqrt{x_N(t)} - 2[\alpha - (1-\gamma) f'(0)]\sqrt{N} x_N(t) \tag{22}$$

Considering also that $x_N(0) = E_N^2$ which is bounded by a constant C (see (5)), we have, by the comparison theorem for differential equations, that $x_N(t) \leq y_N(t)$, where

$$\begin{aligned}\frac{d}{dt} y_N(t) &= A + B\sqrt{y_N(t)} - 2[\alpha - (1-\gamma) f'(0)]\sqrt{N} y_N(t) \\ y_N(0) &= C.\end{aligned}$$

It follows from Lemma B.1 that for every $t > 0$

$$\sup_N N^{\frac{1}{2}} \mathbb{E}\left[\tilde{Z}_N^2(t)\right] < +\infty. \tag{23}$$

From the pointwise estimate (23) we can derive a uniform estimate by using Theorem A.2. First we identify the index n appearing in Theorem A.2 with $N^{\frac{3}{16}}$, so all processes are indexed by N rather than n. Set $X_N(t) = \tilde{Z}_N^2(t)$. Then (18) can be rewritten in the form

$$dX_N(t) = S_N(t)dt + \int_{[0,t]\times\mathcal{Y}} f_N(s^-, y)[\Lambda_N(ds, dy) - A_N(s, dy)ds]$$

with

$$\mathcal{Y} = [0, +\infty) \times \{+, -\}, \quad \Lambda_N(ds, du, \pm) = \sum_{i=1}^N n_i^\pm(\sqrt{N}ds, du), \quad A_N(s, du, \pm) = N^{\frac{3}{2}} du$$

$$f_N(s, u, \pm) = \mathbf{1}_{[0, \tilde{\lambda}^\pm(\sqrt{N}s))}(u)\left[\left(\tilde{Z}_N(s) \pm \frac{1-\gamma}{2\sqrt{N}}\right)^2 - \tilde{Z}_N^2(s)\right], \tag{24}$$

$$S_N(t) = 2F_N \tilde{Z}_N(t) - 2\alpha\sqrt{N}\tilde{Z}_N^2(t). \tag{25}$$

We now verify that conditions in Theorem A.2 hold, with the only difference that the initial time is $\varepsilon > 0$ rather than zero. We set $d = 2$, $\alpha_N = N^{\frac{1}{8}}$, $\beta_N = 1$. With these choices requirements (55) are satisfied. Now, using (23) and the fact that $\tilde{Z}_N$ is bounded (by (15) and the boundedness of $\tilde{Y}_N$), we have

$$\mathbb{E}\left[X_N^d(\varepsilon)\right] = \mathbb{E}\left[\tilde{Z}_N^4(\varepsilon)\right] \leq A\mathbb{E}\left[\tilde{Z}_N^2(\varepsilon)\right] \leq BN^{-\frac{1}{2}} \leq C_1 N^{-\frac{1}{4}} = C_1\alpha_N^{-d}$$

for some constants A, B and C_1, so condition (56) is checked. Condition (57) follows from (25), the fact that $\tilde{Z}_N$ is bounded and that $\sqrt{N} \geq n = N^{\frac{3}{16}}$. Note that no stopping time is needed here. We now consider conditions (58) and (59). Note that, by (24),

$$|f_N(s, u, \pm)| \leq \frac{C}{\sqrt{N}} \leq C_4\alpha_N^{-1}$$

for some constants C and C_4; moreover

$$\begin{aligned}\int_{\mathcal{Y}} (f_N(t, y))^2 A_N(t, dy) =& N^{\frac{3}{2}}\tilde{\lambda}^+(\sqrt{N}t)\left[\left(\tilde{Z}_N(s) + \frac{1-\gamma}{2\sqrt{N}}\right)^2 - \tilde{Z}_N^2(s)\right]^2 \\ &+ N^{\frac{3}{2}}\tilde{\lambda}^-(\sqrt{N}t)\left[\left(\tilde{Z}_N(s) - \frac{1-\gamma}{2\sqrt{N}}\right)^2 - \tilde{Z}_N^2(s)\right]^2 \leq C_5\end{aligned}$$

for some constant C_5, as, for some $C > 0$,

$$\tilde{\lambda}^\pm(\sqrt{N}t) \leq \frac{C}{\sqrt{N}}$$

by definition of $\tilde{\lambda}^{\pm}$ and boundedness of $\tilde{Z}_N$ and $\tilde{Y}_N$, and

$$\left[\left(\tilde{Z}_N(s) \pm \frac{1-\gamma}{2\sqrt{N}}\right)^2 - \tilde{Z}_N^2(s)\right]^2 \leq \frac{C}{N}.$$

Thus all conditions in Theorem A.2 are verified, and we obtain that, for all $\varepsilon > 0$,

$$\sup_{t\in[\varepsilon,T]} \tilde{Z}_N^2(t) \longrightarrow 0 \text{ in probability and in } L^1. \tag{26}$$

We now study Y_N. Using (14) we obtain

$$\begin{aligned}
Y_N(t) &= \frac{\sqrt{N}}{2}\left[\left(\frac{2}{1+\beta}+\frac{2\beta}{1+\beta}\gamma\right)m_N^+(\sqrt{N}t)+\left(\frac{2\beta}{1+\beta}(1-\gamma)+2\beta\gamma\right)m_N^-(\sqrt{N}t)\right]\\
&= \frac{\sqrt{N}}{2}\left[-\alpha\int_0^{\sqrt{N}t}\left(\left(\frac{2}{1+\beta}+\frac{2\beta}{1+\beta}\gamma\right)m_N^+(s)+\left(\frac{2\beta}{1+\beta}(1-\gamma)+2\beta\gamma\right)m_N^-(s)\right)ds\right]\\
&\quad+\frac{\sqrt{N}}{2}\left[\left(\frac{2}{1+\beta}+\frac{2\beta}{1+\beta}\gamma\right)\int_0^{\sqrt{N}t} f\left(\frac{1}{\sqrt{N}}Y_N\left(s/\sqrt{N}\right)+\frac{1}{\sqrt{N}}\frac{2\beta}{1+\beta}Z_N\left(s/\sqrt{N}\right)\right)ds\right.\\
&\quad\left.+\left(\frac{2\beta}{1+\beta}(1-\gamma)+2\beta\gamma\right)\int_0^{\sqrt{N}t} f\left(\frac{1}{\sqrt{N}}Y_N(s/\sqrt{N})-\frac{1}{\sqrt{N}}\frac{2}{1+\beta}Z_N(s/\sqrt{N})\right)ds\right]\\
&\quad+\frac{\sqrt{N}}{2}\left[\left(\frac{2}{1+\beta}+\frac{2\beta}{1+\beta}\gamma\right)\left(a_N^++b_N^+t\right)+\left(\frac{2\beta}{1+\beta}(1-\gamma)+2\beta\gamma\right)\left(a_N^-+b_N^-t\right)\right]\\
&\quad+\frac{\sqrt{N}}{2}\left[\left(\frac{2}{1+\beta}+\frac{2\beta}{1+\beta}\gamma\right)M_N^+(\sqrt{N}t)+\left(\frac{2\beta}{1+\beta}(1-\gamma)+2\beta\gamma\right)M_N^-(\sqrt{N}t)\right]\\
&= C_N + D_N t - \alpha\sqrt{N}\int_0^t Y_N(s)ds\\
&\quad+\frac{N}{2}\left[\left(\frac{2}{1+\beta}+\frac{2\beta}{1+\beta}\gamma\right)\int_0^t f\left(\frac{1}{\sqrt{N}}Y_N(s)+\frac{1}{\sqrt{N}}\frac{2\beta}{1+\beta}Z_N(s)\right)ds\right.\\
&\quad\left.+\left(\frac{2\beta}{1+\beta}(1-\gamma)+2\beta\gamma\right)\int_0^t f\left(\frac{1}{\sqrt{N}}Y_N(s)-\frac{1}{\sqrt{N}}\frac{2}{1+\beta}Z_N(s)\right)ds\right]\\
&\quad+M_N^Y(t)
\end{aligned} \tag{27}$$

where

$$\begin{aligned}
C_N &= \frac{\sqrt{N}}{2}\left[\left(\frac{2}{1+\beta}+\frac{2\beta}{1+\beta}\gamma\right)a_N^++\left(\frac{2\beta}{1+\beta}(1-\gamma)+2\beta\gamma\right)a_N^-\right]\\
D_N &= \frac{\sqrt{N}}{2}\left[\left(\frac{2}{1+\beta}+\frac{2\beta}{1+\beta}\gamma\right)b_N^++\left(\frac{2\beta}{1+\beta}(1-\gamma)+2\beta\gamma\right)b_N^-\right]
\end{aligned}$$

and $M_N^Y(t)$ is a martingale with quadratic variation

$$\begin{aligned}
\sqrt{N}&\left[\left(\frac{1}{1+\beta}+\frac{\beta}{1+\beta}\gamma\right)^2\int_0^t f\left(\frac{1}{\sqrt{N}}Y_N(s)+\frac{1}{\sqrt{N}}\frac{2\beta}{1+\beta}Z_N(s)\right)ds\right.\\
&\qquad\left.+\left(\frac{\beta}{1+\beta}(1-\gamma)+\beta\gamma\right)^2\int_0^t f\left(\frac{1}{\sqrt{N}}Y_N(s)-\frac{1}{\sqrt{N}}\frac{2}{1+\beta}Z_N(s)\right)ds\right].
\end{aligned} \tag{28}$$

We now use the Taylor expansion

$$f(u) = f'(0)u + \frac{1}{2}f''(0)u^2 + R(u)u^3, \tag{29}$$

where $R(u)$ is a bounded function (the boundedness of $R(u)$ follows from the boundedness of the third derivative). After a simple computation, we rewrite (27) as

$$\begin{aligned}
Y_N(t) &= C_N + D_N t + \sqrt{N}\int_0^t [-\alpha + f'(0)(1+\beta\gamma)]Y_N(s)ds \\
&+ \frac{1}{2}(1+\beta\gamma)f''(0)\int_0^t Y_N^2(s)ds + A(\beta,\gamma)f''(0)\int_0^t Y_N(s)Z_N(s)ds + B(\beta,\gamma)f''(0)\int_0^t Z_N^2(s)ds \\
&+ \frac{1}{\sqrt{N}}C(\beta,\gamma)\int_0^t R\left(\frac{1}{\sqrt{N}}Y_N(s) + \frac{1}{\sqrt{N}}\frac{2\beta}{1+\beta}Z_N(s)\right)\left(\frac{1}{\sqrt{N}}Y_N(s) + \frac{1}{\sqrt{N}}\frac{2\beta}{1+\beta}Z_N(s)\right)^3 ds \\
&+ \frac{1}{\sqrt{N}}D(\beta,\gamma)\int_0^t R\left(\frac{1}{\sqrt{N}}Y_N(s) - \frac{1}{\sqrt{N}}\frac{2}{1+\beta}Z_N(s)\right)\left(\frac{1}{\sqrt{N}}Y_N(s) - \frac{1}{\sqrt{N}}\frac{2}{1+\beta}Z_N(s)\right)^3 ds \\
&+ M_N^Y(t)
\end{aligned} \tag{30}$$

where $A(\beta,\gamma)$, $B(\beta,\gamma)$, $C(\beta,\gamma$ and $D(\beta,\gamma)$ are constants depending on β,γ, whose precise value will not be relevant. It should be remarked that in this expansion the term of order $\sqrt{N}\int_0^T Z_N(s)ds$ has zero coefficient: this is the main motivation for the specific choice of the combinators in the definition of Y_N and Z_N.

We are now ready to prove the first part of our asymptotic result.

Theorem 3.1. *Suppose the criticality condition*

$$\alpha = f'(0)(1+\beta\gamma) \tag{31}$$

holds. Then the sequence $(Y_N(t))_{t\in[0,T]}$ converges in distribution to the unique solution of the SDE

$$\begin{aligned}
dy(t) &= \left[\left(\frac{2}{1+\beta} + \frac{2\beta}{1+\beta}\gamma\right)b^+ + \left(\frac{2\beta}{1+\beta}(1-\gamma) + 2\beta\gamma\right)b^- + \frac{1}{2}(1+\beta\gamma)f''(0)y^2(t)\right]dt \\
&+ \frac{\sqrt{f'(0)(1+\beta^2)}(1+\beta\gamma)}{1+\beta}\sqrt{y(t)}dB(t) \\
y(0) &= \left(\frac{2}{1+\beta} + \frac{2\beta}{1+\beta}\gamma\right)a^+ + \left(\frac{2\beta}{1+\beta}(1-\gamma) + 2\beta\gamma\right)a^-,
\end{aligned} \tag{32}$$

where B is a simple Brownian motion.

Proof. We use next Theorem A.1, for the one-dimensional process $Y_N(t)$. We choose

$$\begin{aligned}
B^N(t) &:= D_N t \\
&+ \frac{1}{2}(1+\beta\gamma)f''(0)\int_0^t Y_N^2(s)ds + A(\beta,\gamma)f''(0)\int_0^t Y_N(s)Z_N(s)ds + B(\beta,\gamma)f''(0)\int_0^t Z_N^2(s)ds \\
&+ \frac{1}{\sqrt{N}}C(\beta,\gamma)\int_0^t R\left(\frac{1}{\sqrt{N}}Y_N(s) + \frac{1}{\sqrt{N}}\frac{2\beta}{1+\beta}Z_N(s)\right)\left(\frac{1}{\sqrt{N}}Y_N(s) + \frac{1}{\sqrt{N}}\frac{2\beta}{1+\beta}Z_N(s)\right)^3 ds \\
&+ \frac{1}{\sqrt{N}}D(\beta,\gamma)\int_0^t R\left(\frac{1}{\sqrt{N}}Y_N(s) - \frac{1}{\sqrt{N}}\frac{2}{1+\beta}Z_N(s)\right)\left(\frac{1}{\sqrt{N}}Y_N(s) - \frac{1}{\sqrt{N}}\frac{2}{1+\beta}Z_N(s)\right)^3 ds,
\end{aligned} \tag{33}$$

$$b(x) := \left[\left(\frac{2}{1+\beta} + \frac{2\beta}{1+\beta}\gamma\right)b^+ + \left(\frac{2\beta}{1+\beta}(1-\gamma) + 2\beta\gamma\right)b^- + \frac{1}{2}(1+\beta\gamma)f''(0)x^2\right], \tag{34}$$

$$\begin{aligned}
A^N(t) := \frac{\sqrt{N}}{4}&\left[\left(\frac{2}{1+\beta} + \frac{2\beta}{1+\beta}\gamma\right)^2\int_0^t f\left(\frac{1}{\sqrt{N}}Y_N(s) + \frac{1}{\sqrt{N}}\frac{2\beta}{1+\beta}Z_N(s)\right)ds\right. \\
&\left. + \left(\frac{2\beta}{1+\beta}(1-\gamma) + 2\beta\gamma\right)^2\int_0^t f\left(\frac{1}{\sqrt{N}}Y_N(s) - \frac{1}{\sqrt{N}}\frac{2}{1+\beta}Z_N(s)\right)ds\right],
\end{aligned} \tag{35}$$

$$a(x) = \frac{\alpha(1+\beta^2)(1+\beta\gamma)}{(1+\beta)^2}|x|. \tag{36}$$

By (28) and (30) it follows that $Y_N - A^N$ is a martingale with quadratic variation A^N. Conditions (48), (49) and (50) are trivial, since A^N and B^N are continuous in t and the jumps of Y_N have order $\frac{1}{\sqrt{N}}$. By Theorem B.2 the SDE (32) has a unique solution, and

$$\lim_{N\to+\infty} Y_N(0) = \lim_{N\to+\infty} C_N = y(0).$$

Thus, to complete the proof, we need to establish (51) and (52). We begin by (52). Note that

$$\begin{aligned}
B^N(t) - \int_0^t b(Y_N(s))ds &= \left[D_N - \left(\frac{2}{1+\beta} + \frac{2\beta}{1+\beta}\gamma\right)b^+ - \left(\frac{2\beta}{1+\beta}(1-\gamma) + 2\beta\gamma\right)b^-\right]t \\
&+ A(\beta,\gamma)f''(0)\int_0^t Y_N(s)Z_N(s)ds + B(\beta,\gamma)f''(0)\int_0^t Z_N^2(s)ds \\
&+ \frac{1}{\sqrt{N}}C(\beta,\gamma)\int_0^t R\left(\frac{1}{\sqrt{N}}Y_N(s) + \frac{1}{\sqrt{N}}\frac{2\beta}{1+\beta}Z_N(s)\right)\left(\frac{1}{\sqrt{N}}Y_N(s) + \frac{1}{\sqrt{N}}\frac{2\beta}{1+\beta}Z_N(s)\right)^3 ds \\
&+ \frac{1}{\sqrt{N}}D(\beta,\gamma)\int_0^t R\left(\frac{1}{\sqrt{N}}Y_N(s) - \frac{1}{\sqrt{N}}\frac{2}{1+\beta}Z_N(s)\right)\left(\frac{1}{\sqrt{N}}Y_N(s) - \frac{1}{\sqrt{N}}\frac{2}{1+\beta}Z_N(s)\right)^3 ds
\end{aligned} \tag{37}$$

To prove (52) we show that all terms in the r.h.s of (37) converge to zero in probability after replacing t by $t\wedge\tau_h^N$, with $\tau_h^N := \inf\{t>0 : Y_N(t) > h\}$. The first term in easy, since

$$\lim_{N\to+\infty} D_N = \left(\frac{2}{1+\beta} + \frac{2\beta}{1+\beta}\gamma\right)b^+ + \left(\frac{2\beta}{1+\beta}(1-\gamma) + 2\beta\gamma\right)b^-$$

by (5). The estimate of the next two terms in (37) is based on the facts that, for $t \le \tau_h^N$, we have that $\tilde{Z}_N(t) = Z_N(t)$, $Y_N(t) \le h$ and, by (15)

$$|Z_N(t)| \le \frac{1+\beta}{2}Y_N(t) \le \frac{1+\beta}{2}h.$$

Thus

$$\left|A(\beta,\gamma)f''(0)\int_0^{t\wedge\tau_h^N} Y_N(s)Z_N(s)ds + B(\beta,\gamma)f''(0)\int_0^{t\wedge\tau_h^N} Z_N^2(s)ds\right| \le \text{constant}\cdot\int_0^t \tilde{Z}_N^2(s)\wedge\frac{(1+\beta)^2}{4}h^2 ds$$

which, by (23) and dominated convergence, converges to zero in L^1 and so in probability. For the remaining two terms in (37) we use again

$$|Z_N(t)| \le \frac{1+\beta}{2}Y_N(t),$$

which implies that

$$\int_0^{t\wedge\tau_h^N} R\left(\frac{1}{\sqrt{N}}Y_N(s) + \frac{1}{\sqrt{N}}\frac{2\beta}{1+\beta}Z_N(s)\right)\left(\frac{1}{\sqrt{N}}Y_N(s) + \frac{1}{\sqrt{N}}\frac{2\beta}{1+\beta}Z_N(s)\right)^3 ds$$

is bounded by a constant. This completes the proof on (52). The proof of (51) is similar but easier, as it requires only a first order expansion for f ∎

3.3 Convergence of the joint price-volatility process

Let us recall the rescaled log-price in (6)

$$\Pi_N(t) = \frac{1}{\sqrt{N}}\sum_{i=1}^{N} P_i(\sqrt{N}t) = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[\int_{[0,\sqrt{N}t]\times[0,+\infty)} \mathbf{1}_{[0,\lambda_N^+(s-))}(u) n_i^+(ds,du) - \int_{[0,\sqrt{N}t]\times[0,+\infty)} \mathbf{1}_{[0,\lambda_N^-(s-))}(u) n_i^-(ds,du)\right].$$

Compensating as above the Poisson Processes we obtain

$$\begin{aligned}\Pi_N(t) = N\Bigg[&\int_0^t f\left(\frac{1}{\sqrt{N}}Y_N(s) + \frac{1}{\sqrt{N}}\frac{2\beta}{1+\beta}Z_N(s)\right)ds \\ &- \int_0^t f\left(\frac{1}{\sqrt{N}}Y_N(s) - \frac{1}{\sqrt{N}}\frac{2}{1+\beta}Z_N(s)\right)ds\Bigg] + M_N^{\Pi}(t),\end{aligned} \tag{38}$$

where M_N^{Π} is a martingale with zero mean and quadratic variation

$$\sqrt{N}\left[\int_0^t f\left(\frac{1}{\sqrt{N}}Y_N(s) + \frac{1}{\sqrt{N}}\frac{2\beta}{1+\beta}Z_N(s)\right)ds + \int_0^t f\left(\frac{1}{\sqrt{N}}Y_N(s) - \frac{1}{\sqrt{N}}\frac{2}{1+\beta}Z_N(s)\right)ds\right]. \tag{39}$$

We can also easily identify the covariation between the martingales M_N^{Π} and M_N^{Y}, which is given by

$$\begin{aligned}\sqrt{N}\Bigg[&\left(\frac{1}{1+\beta} + \frac{\beta}{1+\beta}\gamma\right)\int_0^t f\left(\frac{1}{\sqrt{N}}Y_N(s) + \frac{1}{\sqrt{N}}\frac{2\beta}{1+\beta}Z_N(s)\right)ds \\ &- \left(\frac{\beta}{1+\beta}(1-\gamma) + \beta\gamma\right)\int_0^t f\left(\frac{1}{\sqrt{N}}Y_N(s) - \frac{1}{\sqrt{N}}\frac{2}{1+\beta}Z_N(s)\right)ds\Bigg].\end{aligned} \tag{40}$$

We can now rewrite (38) by using the Taylor expansion (29), and we obtain

$$\begin{aligned}\Pi_N(t) &= 2f'(0)\sqrt{N}\int_0^t Z_N(s)ds \\ &\quad + \frac{1}{2}f''(0)\int_0^t\left[\left(Y_N(s) + \frac{2\beta}{1+\beta}Z_N(s)\right)^2 - \left(Y_N(s) - \frac{2}{1+\beta}Z_N(s)\right)^2\right]ds \\ &\quad + \frac{1}{\sqrt{N}}\int_0^t R\left(Y_N(s) + \frac{2\beta}{1+\beta}Z_N(s)\right)\left(Y_N(s) + \frac{2\beta}{1+\beta}Z_N(s)\right)^3 ds \\ &\quad + \frac{1}{\sqrt{N}}\int_0^t R\left(Y_N(s) - \frac{2}{1+\beta}Z_N(s)\right)\left(Y_N(s) - \frac{2}{1+\beta}Z_N(s)\right)^3 ds + M_N^{\Pi}(t) \\ &= 2f'(0)\sqrt{N}\int_0^t Z_N(s)ds \\ &\quad + f''(0)\int_0^t\left[2Y_N(s)Z_N(s) + 2\frac{\beta-1}{\beta+1}Z_N^2(s)\right]ds \\ &\quad + \frac{1}{\sqrt{N}}\int_0^t R\left(Y_N(s) + \frac{2\beta}{1+\beta}Z_N(s)\right)\left(Y_N(s) + \frac{2\beta}{1+\beta}Z_N(s)\right)^3 ds \\ &\quad + \frac{1}{\sqrt{N}}\int_0^t R\left(Y_N(s) - \frac{2}{1+\beta}Z_N(s)\right)\left(Y_N(s) - \frac{2}{1+\beta}Z_N(s)\right)^3 ds + M_N^{\Pi}(t).\end{aligned} \tag{41}$$

The problem here is in the term $2f'(0)\sqrt{N}\int_0^t Z_N(s)ds$. We have shown that

$$\lim_{N\to+\infty}\mathbb{E}\left[\int_0^{T\wedge\tau_h^N} Z_N^2(s)ds\right] = 0,$$

but this does not allow to control $\sqrt{N}\int_0^t Z_N(s)ds$. To overcome this difficulty, we go back to (16), which gives

$$\begin{aligned}
Z_N(t) = E_N + F_N t - \alpha\sqrt{N}\int_0^t Z_N(s)ds \\
+ \frac{1-\gamma}{2\sqrt{N}}\int_0^t \Bigg[f\left(\frac{1}{\sqrt{N}}Y_N\left(s/\sqrt{N}\right) + \frac{1}{\sqrt{N}}\frac{2\beta}{1+\beta}Z_N\left(s/\sqrt{N}\right)\right) \\
- f\left(\frac{1}{\sqrt{N}}Y_N(s/\sqrt{N}) - \frac{1}{\sqrt{N}}\frac{2}{1+\beta}Z_N(s/\sqrt{N})\right)\Bigg] ds + M_N^Z(t),
\end{aligned}$$

where $M_N^Z(t) = \frac{1-\gamma}{2}M_N^\Pi(t)$. Expanding f as above we obtain

$$\begin{aligned}
Z_N(t) = E_N + F_N t - (\alpha - (1-\gamma)f'(0))\sqrt{N}\int_0^t Z_N(s)ds \\
+ (1-\gamma)f''(0)\int_0^t \left[Y_N(s)Z_N(s) + \frac{\beta-1}{1+\beta}Z_N^2(s)\right] ds + Q_N(t) + \frac{1-\gamma}{2}M_N^\Pi(t),
\end{aligned} \tag{42}$$

where $Q_N(t)$ comes from the cubic part of the expansion, and $Q_N(t\wedge\tau_h^N)$ converges uniformly to zero as $N \to +\infty$. We can now use (42) to eliminate the divergent term from (41).

$$\begin{aligned}
\Pi_N(t) = -\frac{2f'(0)}{\alpha-(1-\gamma)f'(0)}Z_N(t) + \frac{2f'(0)}{\alpha-(1-\gamma)f'(0)}E_N + \frac{2f'(0)}{\alpha-(1-\gamma)f'(0)}F_N t \\
+ 2f''(0)\left(2 + \frac{2f'(0)(1-\gamma)}{\alpha-(1-\gamma)f'(0)}\right)\int_0^t \left[Y_N(s)Z_N(s) + \frac{\beta-1}{1+\beta}Z_N^2(s)\right] ds \\
+ \left(1 + \frac{f'(0)(1-\gamma)}{\alpha-(1-\gamma)f'(0)}\right)M_N^\Pi(t) + Q_N(t),
\end{aligned} \tag{43}$$

where $Q_N(t)$ is the same as the one in (42) up to a multiplicative constant.

We can now describe the asymptotic behavior of the log-price process. In what follows by convergence in distribution in $(0,T]$ we mean that the convergence occurs in $[\varepsilon, T]$ for all $\varepsilon > 0$.

Theorem 3.2. *Suppose the criticality condition*

$$\alpha = f'(0)(1+\beta\gamma) \tag{44}$$

holds. The pair (Π_N, Y_N) converges in distribution in $(0,T]$ to the process (π, y) where $y(t)$ solves equation (32), and

$$\begin{aligned}
d\pi(t) &= \frac{1-\gamma}{\gamma(1+\beta)}(b^+ - b^-)dt + \frac{1+\beta\gamma}{\gamma(1+\beta)}\sqrt{2f'(0)y(t)}dW(t) \\
\pi(0) &= \frac{1-\gamma}{\gamma(1+\beta)}(a^+ - a^-),
\end{aligned} \tag{45}$$

where W is a Brownian motion, and

$$d\langle B, W\rangle_t = -\frac{\gamma(\beta^2-1)}{(1+\beta\gamma)\sqrt{2(1+\beta^2)}}dt. \tag{46}$$

Proof. We use Theorem A.1 for the two dimensional process (Π_N, Y_N). The convergence of Y_N has been already established. So, we need to show the convergence of: the drift part of Π_N, the quadratic variation of the martingale part of Π_N, and the quadratic covariation of the two martingale components.

The drift part of Π_N contains the term

$$\frac{2f'(0)}{\alpha-(1-\gamma)f'(0)}F_N t$$

which asymptotically produces the constant drift

$$\frac{f'(0)(1-\gamma)}{\alpha-(1-\gamma)f'(0)}(b^+-b^-).$$

To complete the treatment of the drift we need to show that, for every $\varepsilon, \delta > 0$,

$$\lim_{N\to+\infty} \mathbb{P}\left(\sup_{\varepsilon\le t\le T\wedge\tau_h^N}\left|\int_0^t \left[Y_N(s)Z_N(s)+\frac{\beta-1}{1+\beta}Z_N^2(s)\right]ds\right|\ge\delta\right)=0,$$

which readily follows from (26) and the fact that Y_N is bounded up to time τ_h^N.

The martingale component of Π_N is given by, using $\alpha = f'(0)(1+\beta\gamma)$,

$$\left(1+\frac{f'(0)(1-\gamma)}{\alpha-(1-\gamma)f'(0)}\right)M_N^\Pi(t)=\frac{1+\beta\gamma}{\gamma(1+\beta)}M_N^\Pi(t).$$

Thus, using (39), we need to show that

$$\lim_{N\to+\infty} \mathbb{P}\left(\sup_{\varepsilon\le t\le T\wedge\tau_h^N}\left|\int_0^t A_N(s)ds-\int_0^t 2f'(0)Y_N(s)ds\right|\ge\delta\right)=0, \tag{47}$$

where

$$A_N(s) := \sqrt{N}\left[f\left(\frac{1}{\sqrt{N}}Y_N(s)-\frac{1}{\sqrt{N}}\frac{2}{1+\beta}Z_N(s)\right)+f\left(\frac{1}{\sqrt{N}}Y_N(s)-\frac{1}{\sqrt{N}}\frac{2}{1+\beta}Z_N(s)\right)\right],$$

which is established by a simple first order Taylor expansion of f.

Finally we need to verify the convergence of the covariation. Using (40), it follows that the covariation $\langle M_N^\Pi, M_Y^N\rangle(t)$ converges, in the sense of (51), to

$$f'(0)\frac{(1-\beta)(1+\beta\gamma)}{1+\beta}\int_0^t y(s)ds.$$

Thus, comparing (32), and (45), we need

$$\frac{\sqrt{f'(0)y(t)(1+\beta^2)}(1+\beta\gamma)}{1+\beta}\frac{1+\beta\gamma}{\gamma(1+\beta)}\sqrt{2f'(0)y(t)}d\langle B,W\rangle(t)=f'(0)\frac{(1-\beta)(1+\beta\gamma)}{1+\beta}y(t)dt,$$

which implies (46). ∎

A Useful results

These theorems correspond to Theorem VII, 4.1 [27] and the Proposition in Appendix A in [17].

Theorem A.1 (Diffusion approximation). *Let $(X^N)_{N\in\mathbb{N}}$ and $(B^N)_{N\in\mathbb{N}}$ $\mathbb{R}^d$-valued processes with càdlàg sample paths and let $A^N = (A_{ij}^N)$ be a symmetric $d\times d$ matrix-valued process such that A_{ij}^N has càdlàg sample paths in $\mathbb{R}$ and $A^N(t)-A^N(s)$ is non-negative definite for all $t > s \ge 0$. Let $\mathcal{F}_t^N := \sigma\left(X^N(s), B^N(s), A^N(s) : s\le t\right)$. Let $\tau_h^N := \inf\left\{t>0 : |X^N(t)|\ge h \, or |X^N(t^-)|\ge h\right\}$, where $X^N(t^-) := \lim_{s\to t^-} X^N(s)$ denotes the left-hand limit. Assume that*

- *$M^N := X^N - B^N$ and $M_i^N M_j^N - A_{ij}^N$ $i,j = 1,\dots,d$ are $(\mathcal{F}_t^N)$-local martingales*
- *for each $T>0$, $h>0$*

$$\lim_{N\to+\infty}\mathbb{E}\left[\sup_{t\le T\wedge\tau_h^N}|B^N(t)-B^N(t^-)|^2\right]=0 \tag{48}$$

- *for each $T > 0$, $h > 0$, $i, j = 1, \ldots, d$*

$$\lim_{N \to +\infty} \mathbb{E}\left[\sup_{t \le T \wedge \tau_h^N} |A_{ij}^N(t) - A_{ij}^N(t^-)|\right] = 0 \tag{49}$$

- *for each $T > 0$, $h > 0$*

$$\lim_{N \to +\infty} \mathbb{E}\left[\sup_{t \le T \wedge \tau_h^N} |X^N(t) - X^N(t^-)|^2\right] = 0 \tag{50}$$

- *there exist a continuous, symmetric, non-negative definite $d \times d$ matrix-valued function on $\mathbb{R}^d$, $a = (a_{ij})$, and a continuous function $b : \mathbb{R}^d \to \mathbb{R}^d$ such that, for each $h > 0$, $T > 0$ and $i, j = 1, \ldots, d$, and for all $\epsilon > 0$,*

$$\lim_{N \to +\infty} \mathbb{P}\left(\sup_{t \le T \wedge \tau_h^N} \left|A_{ij}^N(t) - \int_0^t a_{ij}(X^N(s))ds\right| > \epsilon\right) = 0 \tag{51}$$

and

$$\lim_{N \to +\infty} \mathbb{P}\left(\sup_{t \le T \wedge \tau_h^N} \left|B_i^N(t) - \int_0^t b_i(X^N(s))ds\right| > \epsilon\right) = 0 \tag{52}$$

- *the $\mathcal{C}_{\mathbb{R}^d}([0, +\infty))$ martingale problem for*

$$\tilde{A} \coloneqq \left\{ \left(f, Gf \coloneqq \frac{1}{2}\sum_{i,j} a_{ij}\partial_i\partial_j f + \sum_i b_i \partial_i f \right) \ : \ f \in \mathcal{C}_c^\infty\left(\mathbb{R}^d\right) \right\} \tag{53}$$

is well-posed.

- *the sequence of the initial laws of the X^Ns converges in distribution to some probability distribution on $\mathbb{R}^d$, ν.*

Then $\left(X^N\right)_N$ converges in distribution to the solution of the martingale problem for $(\tilde{A}, \nu)$. That is, the laws of the processes $\left(X^N\right)_N$ converge weakly to the law of a process X which is a weak solution of the SDE

$$dX(t) = b(X(t))dt + \Sigma(X(t))dW(t) \tag{54}$$

where $b = (b_i)_i$ and $\left(\Sigma\Sigma^T\right)_{ij} = a_{ij}$ are the drift vector and the diffusion coefficient in (53).

Theorem A.2. *Let $\{X_n(t)\}_{n \ge 1}$ be a sequence of positive semimartingales on a probability space $(\Omega, \mathcal{A}, \mathcal{P})$, with*

$$dX_n(t) = S_n(t)dt + \int_{[0,t] \times \mathcal{Y}} f_n(s^-, y)[\Lambda_n(ds, dy) - A_n(s, dy)ds]$$

Here, Λ_n is a Point Process of intensity $A_n(t, dy)dt$ on $\mathbb{R}^+ \times \mathcal{Y}$, where $\mathcal{Y}$ is a measurable space, and $S_n(t)$ and $f_n(t)$ are $\mathcal{A}_t$-adapted processes, if we consider $(\mathcal{A}_t)_{t \ge 0}$ a filtration on $(\Omega, \mathcal{A}, \mathcal{P})$ generated by Λ_n. Let $d > 1$ and C_i constants independent of n and t. Suppose there exist $\{\alpha_n\}_{n \ge 1}$ and $\{\beta_n\}_{n \ge 1}$, increasing sequences with

$$n^{1/d}\alpha_n^{-1} \xrightarrow{n \to +\infty} 0, n^{-1}\alpha_n \xrightarrow{n \to +\infty} 0, n^{-1}\beta_n \xrightarrow{n \to +\infty} 0 \tag{55}$$

and

$$E\left[\Big(X_n(0)\Big)^d\right] \le C_1 \alpha_n^{-d} \qquad \textit{for all } n \tag{56}$$

Furthermore, let $\{\tau_n\}_{n \ge 1}$ be stopping times such that for $t \in [0, \tau_n]$ and $n \ge 1$,

$$S_n(t) \le -n\delta X_n(t) + \beta_n C_2 + C_3 \qquad \textit{with } \delta > 0 \tag{57}$$

$$\sup_{\omega\in\Omega, y\in\mathcal{Y}, t\le\tau_n} |f_n(t,y)| \le C_4\alpha_n^{-1}, \tag{58}$$

and also assume

$$\int_{\mathcal{Y}} (f_n(t,y))^2 A_n(t,dy) \le C_5\,. \tag{59}$$

Then, for any $\varepsilon > 0$, there exist $C_6 > 0$ and n_0 such that

$$\sup_{n\ge n_0} \mathcal{P}\left\{ \sup_{0\le t\le T\wedge\tau_n} X_n(t) > C_6(n^{1/d}\alpha_n^{-1} \vee \alpha_n n^{-1}) \right\} \le \varepsilon \tag{60}$$

B Auxiliary computations

Lemma B.1. *For $m > 0$ let $y_m(t)$ be the solution of*

$$\begin{aligned} \dot{y}_m(t) &= A + B\sqrt{y_m(t)} - m y_m(t) \\ y_m(0) &= C, \end{aligned}$$

where A, B, C are positive constants. Then for every $t > 0$

$$\limsup_{m\to\infty} m\, y_m(t) < +\infty.$$

Proof. By comparison principle with the solution of the linear ODE one can see that $y_m(t) > 0$. Let $(\bar{Y}_m)_m$ be the sequence $\bar{Y}_m = 4(B^2/m^2 + A/m)$, for which it holds that that $A + B\sqrt{z} - mz < -(m/2)z$ for $z > \bar{Y}_m$. Then $\dot{y}_m < -(m/2)y_m$ when $y_m > \bar{Y}_m$, and $\dot{y}_m < 0$ since $y_m > 0$. Let $\tau_m = \inf_{t>0}\{y_m(t) \le \bar{Y}_m\}$. Consider equation $\dot{z} = -(m/2)z$, whose solution is $z(t) = z(0)e^{-mt/2}$. So $z(2/\sqrt{m}) = z(0)e^{-\sqrt{m}}$. Now, between 0 and $(2/\sqrt{m})\wedge\tau_m$, we have that $\dot{y}_m < -(m/2)y_m$, so by comparison $y_m(2/\sqrt{m}) \le Ce^{-\sqrt{m}} + \bar{Y}_m$. For $t > 2/\sqrt{m}$, we have $y_m(t) \le Ce^{-\sqrt{m}} + \bar{Y}_m$ and $2/\sqrt{m} \to 0$ and $\limsup_{m\to+\infty} m(Ce^{-\sqrt{m}} + \bar{Y}_m) < +\infty$. This implies the statement. ∎

Theorem B.2. *The stochastic equation*

$$dX_t = \sqrt{2aX_t}dW_t + (-bX_t^2 + c)dt, \quad X_0 = x_0 > 0, \tag{61}$$

with $a, b, c > 0$ admits a weak solution for which pathwise uniqueness holds. If $0 < c < a$, the point 0 is attainable and the process is reflected upwards at 0, so that X spends 0 time at 0. If $c \ge a$, the point 0 is not attainable and the process is always strictly positive.

Moreover, the bi-dimensional equation

$$\begin{aligned} dX_t &= \sqrt{2aX_t}dB_t + (-bX_t^2 + c)dt, \quad X_0 = x_0 > 0, \\ d\xi_t &= wdt + v\sqrt{X_t}dW_t, \quad \xi_0 = \bar{\xi}_0 \in \mathbb{R} \end{aligned} \tag{62}$$

with $a, b, c, v > 0$, $w \in \mathbb{R}$, $\langle B, W\rangle = \rho \in [-1,1]$ also admits a weak solution, for which pathwise uniqueness holds

Proof. Let us write $\sigma^2(x) = 2ax$ and $\mu(x) = -bx^2 + c$. Following [37][p.446, Chapter 3. Some results on one-dimensional diffusion processes], we prove that there exist a (weak) unique (pathwise) solution to (61). The diffusion and drift coefficients are of class C_1 on $(0, +\infty)$, the diffusion coefficient is positive on $(0, +\infty)$. So Equation (61) is uniquely defined until the stopping time $T_e = T_0 \wedge T_\infty$ where T_0 is the hitting time of 0 and T_∞ the explosion time. Let us now introduce, relative to (61), *speed measure* M with density m and *scale function* S given by

$$m(x) = \frac{2}{\sigma^2(x)s(x)} \text{ and } S(x) = \int_1^x s(y)dy, \tag{63}$$

with $s(y) = \exp(-\int_1^x 2\mu(y)/\sigma^2(y)dy)$. We have that $S(+\infty) = +\infty$ for any parameters choice, so the process does not explode, and $S(0) = -\infty$ if $c \ge a$, so in this case [37][Theorem 3.1] tells us that

$P(T_e = +\infty) = 1$, so there exist a (weak) unique (pathwise) solution to (61) and with probability one the process remains strictly positive for all times t (the boundary is not attainable). If $0 < c < a$, we have $S(0) \in \mathbb{R}$ and the process hits 0, i.e. $P(T_0 < +\infty) = 1$. We have $dX_t = cdt$, with $c > 0$, when $X_t = 0$, so the process is reflected at 0 and the explosion time is still infinity, so existence still holds. Pathwise uniqueness can be shown adapting [37][Theorem 3.2], using the fact that $-bX^2 < 0$ to control I_1 in the proof. Note that this is consistent with the known results on CIR process.

To obtain a solution to the bi-dimensional equation, the second component must be given by

$$\xi_t = \xi_0 + wt + v\int_0^t \sqrt{X_s}dW_s$$

To have weak existence and pathwise uniqueness, we check that $\sqrt{X_\cdot} \in M^2[0,T]$.

$$E\int_0^t (\sqrt{X_s})^2 ds = \int_0^t E[X_s]ds < +\infty$$

by comparison with a suitable CIR process (or computing it using speed measure and scale function). Therefore the statement holds true. ∎

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